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On solitary wave solutions of ac-driven complex Ginzburg–Landau equation

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Abstract

A new class of periodic solutions of modified complex Ginzburg–Landau equation phase locked to a time-dependent force, by applying a nonfeedback mechanism for chaos control, have been found. The reported solutions are *necessarily* of the rational form containing trigonometric and hyperbolic functions.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Much attention has been paid to the study of the most ubiquitous spatiotemporal (ST) chaos prevalent in the equations of nonlinear dissipative systems, in the recent past. Although chaos may be advantageous in some situations, in many other situations it should be avoided or controlled [1–4]. Various techniques such as different feedback mechanisms [1, 5] and nonfeedback control mechanisms [4, 6, 7, 9–13], have been devised in that order for suppressing chaos and converting a chaotic behaviour into a desired regular one, as well. For example, the problem of suppressing chaos by harmonic (or periodic) perturbations has become the object of intensive study in recent years. Geometric resonance (GR), an elegant route to eliminate chaos [14], falls in that category. As the periodic perturbation methods suffer from the deficiency of lack of common concept for constructing appropriate perturbations, which direct the trajectory to the target, the method of appropriate nonfeedback control mechanism comes in handy to convert the chaotic behaviour into the desired ones. In [14, 15], González *et al* used the concept of geometric resonance as a method of chaos control to a very general class of ST chaos exhibited by the sine-Gordon, nonlinear Schrödinger, Boussinesq, Toda lattice and complex Ginzburg–Landau equations, when these are nonintegrable. GR is an extension of the linear-system-based notion of resonance to a fully nonlinear formulation based on a local energy conservation requirement [7].

The recently formulated concept of GR [7] provides a mechanism for nonfeedback control of chaos. This has been delineated in the present work that deals with another equation of the Ginzburg–Landau-type with an ac driver. We have undertaken the study of eliminating chaos by using an appropriate control signal $F_c(x, t)$ in this modified CGLE that satisfies the GR condition. At GR the amplitude, frequency and space-time shape of a very general driving force must satisfy some conditions so that some dynamical properties of the conservative system are preserved. We will call that solution a GR solution of nonintegrable equation. This implies a local energy conservation requirement. The energy integral that is conserved for the integrable Hamiltonian system is locally conserved for the full nonintegrable equation if the GR condition holds. We can use this condition as a mechanism for chaos control when an additional condition holds: the GR solution must be an asymptotically stable solution of the (full) equation.

Let us assume for the optimal choice of the suppressory driving term, and its phase, the corresponding actual solution $x(t)$ remains—after the transient—close to the GR solution: $x(t) = x_{GR} + \delta x(t)$, where $\delta x(t)$ is a small deviation with $\mathrm{d}(\delta x)/\mathrm{d}t \ll \delta x/T'$. By considering the energy of the system as a 'local almost adiabatic invariant' [8], we write an approximate GR condition

$$\left\langle \frac{\mathrm{d}H}{\mathrm{d}t} \right\rangle_{T'} \simeq 0, \tag{1}$$

where H is the energy of the system and T' is the period of the solution of the integrable Hamiltonian system.

2. The ac-driven CGLE

Also, spiral waves in the CGLE with a time-dependent periodic external force has been studied [16]. Despite its simplicity, the forced CGLE, depending on several factors, such as the spatial dimension, the mode of the frequency locking and the behaviour of the corresponding unforced system, describes a large variety of phenomena [17–24]. For a sufficiently large forcing amplitude, a homogeneous with no spatial structure has been observed in [25]. We are interested here in the modified CGLE [26, 27] with an ac driver, in the form of a travelling wave

$$\phi_t = \phi + (1 + \mathrm{i}c_1)\phi_{xx} - (1 - \mathrm{i}c_3)|\phi|^2\phi + F_c(x, t) - \mathrm{i}\varepsilon_1\,\mathrm{e}^{\mathrm{i}\omega t}, \tag{2}$$

where the term $F_c(x, t)$ is the control signal and $\varepsilon_1\,\mathrm{e}^{\mathrm{i}\omega t}$ is the time-dependent external driver. Here ε_1 is the amplitude of the driver, and ω stands for its frequency. Without the control signal and the driver, the turbulence develops when the Benjamin–Feir condition $1 - c_1c_3 < 0$ is satisfied. Near the unstable side of the stability border phase turbulence is observed. Now we rewrite equation (2) in the following form, for suppression of ST chaos:

$$\mathrm{i}\phi_t + c_1\phi_{xx} + c_3|\phi|^2\phi = \mathrm{i}(\phi_{xx} + \phi - |\phi|^2\phi) + \mathrm{i}F_c(x, t) + \varepsilon_1\,\mathrm{e}^{\mathrm{i}\omega t}. \tag{3}$$

When the right-hand side of equation (3) is zero, it reduces to the nonlinear Schrödinger equation (NLSE). If $\phi(x, t) = f(x)\,\mathrm{e}^{\mathrm{i}\omega t}$ is a soliton solution of NLSE phase locked with a source, then we can use the following controlling signal:

$$F_c(x, t) = [f^3(x) - f(x) - f_{xx}(x) - \mathrm{i}\varepsilon_2]\,\mathrm{e}^{\mathrm{i}\omega t}. \tag{4}$$

Equation (3) (with $F_c = 0$, and without a driver) presents phase turbulence for $c_1 = 2$, $c_3 = 0.8$. We can suppress the turbulence exhibited by equation (3) using $F_c(x, t)$ given by equation (4) with $\omega = 12$ and $f(x)$ is the one-soliton solution of the following ODE:

$$c_1 f_{xx} - \omega f + c_3 f^3 - \varepsilon = 0. \tag{5}$$

The fact that equation (4) has been taken that satisfies the GR condition provided $\varepsilon = \varepsilon_1 + \varepsilon_2$. Thus in equation (3), if $F_c(x, t)$ given by equation (4), the function $\phi(x, t)$ will emerge as the exact solution of the complete equation.

3. Exact solutions

Our goal in this paper is to present the newly found periodic solutions of equation (5) expressed as rational form in terms of elliptic functions. To accomplish this we start with a fractional transformation (FT) [28]

$$f(x) = \frac{A + Bq^2(x)}{1 + Dq^2(x)}, \tag{6}$$

where the determinant $AD - B \neq 0$. This FT connects the solutions of the modified CGLE with a source to the elliptic equation $q'' \pm aq \pm \lambda q^3 = 0$, under the influence of a control signal. Then it is clear that the coefficients of $q^n(x)$, for $n = 0, 2, 4, 6$ can be set to zero to reduce the problem to an algebraic one, and obtain the solutions. In getting the algebraic equations, use has been made of the following relations for various derivatives of q: $q'' = q + q^3$ and $q'^2 = \frac{1}{2}q^4 + 2E_0$, where E_0 is the integration constant, for $a = \lambda = 1$. Furthermore, it is asserted that q can be taken as any of the three Jacobi elliptic functions with an appropriate modulus parameter: $\mathrm{cn}(x, m)$, $\mathrm{dn}(x, m)$ and $\mathrm{sn}(x, m)$. For definiteness, we start with the assumption $q = \mathrm{cn}$ evidently, the coefficients of $\mathrm{cn}^n(x, m)$, for $n = 0, 2, 4, 6$ can be set to zero, and thereby obtain the four consistency equations. The identities satisfied by the cnoidal functions make them amenable for finding the exact solutions of the nonlinear ODE of the form described by equation (5). In simplifying the second derivative of q, we used the following important identities satisfied by the cnoidal functions:

$$\begin{aligned}
&\mathrm{cn}^2\,\mathrm{sn}^2\,\mathrm{dn}^2 = \mathrm{cn}^2(1-m) + (2m-1)\mathrm{cn}^4 - m\,\mathrm{cn}^6\\
&\mathrm{cn}^4\,\mathrm{dn}^2 - m\,\mathrm{cn}^4\,\mathrm{sn}^2 = \mathrm{cn}^4(1-2m) + 2m\,\mathrm{cn}^6\\
&\mathrm{cn}^2\,\mathrm{dn}^2 - \mathrm{sn}^2\,\mathrm{dn}^2 - m\,\mathrm{cn}^2\,\mathrm{sn}^2 = 2\mathrm{cn}^2(1-2m) + 3m\,\mathrm{cn}^4 + m - 1.
\end{aligned} \tag{7}$$

The four consistency conditions are

$$-\omega A - 2c_1(AD - B)(1-m) + c_3A^3 - \varepsilon = 0, \tag{8}$$

$$-2\omega AD - \omega B + 6c_1(AD - B)D(1-m) - 4c_1(AD - B)(2m-1) + 3c_3A^2B - 3\varepsilon D = 0, \tag{9}$$

$$-A\omega D^2 - 2\omega BD + 4c_1(AD - B)D(2m-1) + 6c_1(AD - B)m + 3c_3AB^2 - 3\varepsilon D^2 = 0, \tag{10}$$

$$-\omega BD^2 - 2c_1(AD - B)Dm + c_3B^3 - \varepsilon D^3 = 0. \tag{11}$$

For example, if one considers the solution of equation (5) in terms of dn function, then it yields a constant background solution for $m = 0$. Under special conditions, very interesting solutions are obtained for equation (5). Thus, we shall present them here, with the specifications for the regimes in which they apply, with the new rational solutions we found. The three solutions are presented in the order that appears to be most natural, for various limiting values of the modulus parameter of the cnoidal function.

Case I (trigonometric solution). In the consistency conditions (equations (8)–(11)) if we put $A = 0$ and $m = 0$, we find that a periodic solution of the following type emerges:

$$f(x) = (\varepsilon/2c_1)\frac{\cos^2(x)}{1 - (2/3)\cos^2(x)}, \tag{12}$$

where $c_1 = \frac{\omega}{4}$ and $c_3 = \frac{2\omega^3}{27\epsilon^2}$. This is a non-singular solution.

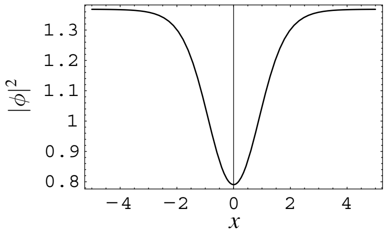

Figure 1. Plot depicting the intensity of the dark solitary wave solution for $\varepsilon = -1$ and $\omega = 0.18$.

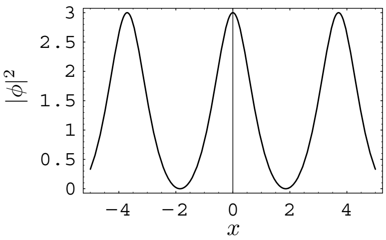

Figure 2. Plot depicting the intensity of the cnoidal wave solution for $\varepsilon = 1$, $c_1 = 0.5$ and $\omega = 5$.

Case II (hyperbolic solution). In order to obtain solitary wave solutions, we impose the following conditions: $B = 0$ and $m = 1$. This yields

$$f(x) = (-3\varepsilon/\omega)\frac{1}{1-(3/2)\,\mathrm{sech}^2(x)}. \tag{13}$$

This is a singular solution. If we consider the case, $AD = 1$ and $B = 0$, then we get a non-singular, hyperbolic solution;

$$f(x) = \left(\frac{-3\varepsilon}{2(\omega+2c_1)}\right)\frac{1}{1+D\,\mathrm{sech}^2(x)}, \tag{14}$$

where $D = \frac{-2(\omega+2c_1)}{3\varepsilon}$, $c_1 = \frac{-(\omega+\frac{9\varepsilon}{8})\pm\frac{3\sqrt{\varepsilon}}{2}\sqrt{\omega+\frac{9\varepsilon}{16}}}{2}$ and $c_3 = \frac{4}{27\varepsilon^2}\left(\omega^3-12\omega c_1^2-16c_1^3\right)$. After simple manipulations, the above solution can be transformed to already reported results [29, 30]. This dark solitary wave solution has been depicted in figure 1 for the parameter values specified in the figure caption.

Although we have not noted here, if we choose A, B and D non-zero, for $m = 1$, we obtain a bright solitary wave.

Case III (pure cnoidal solutions). Now we obtain a cnoidal solution, for $m = 1/2$ and $A = 0$,

$$f(x) = (\varepsilon/c_1)\frac{\mathrm{cn}^2(x,m)}{1+D\,\mathrm{cn}^2(x,m)}, \tag{15}$$

where $D = -\frac{\omega}{6c_1}$ and $c_3 = \left(\frac{1}{\epsilon^2}\right)\left(\frac{5\omega^3}{216}+\frac{c_1^2\omega}{6}\right)$. We further emphasize that this cnoidal solution is a non-singular solution for $\omega < 6c_1$, as depicted in figure 2.

At this point, we emphasize that the following cases are forbidden due to the fact that the source becomes a vanishing one, as the presence of the source in equation (3) is pivotal for the existence of the Lorentzian/rational-type solutions. For $m = 0$, $A = 0$ is not allowed, and for $m = 1$, neither $A = 0$ nor $D = 0$ is allowed.

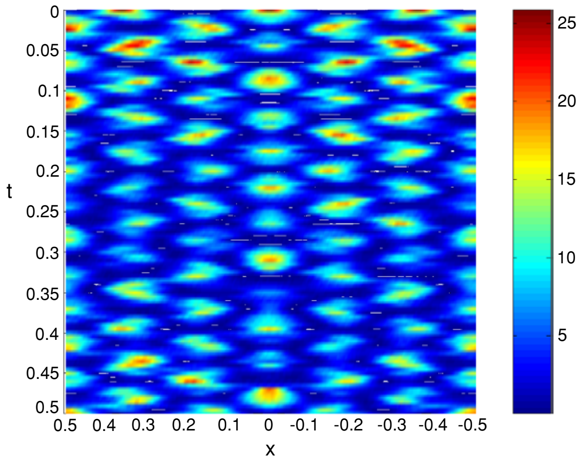

Figure 3. Regular dynamics obtained when the controlling signal $F_c(x, t)$ is used for $\omega = 12$.

Since the localized solitons are usually robust, we have performed numerical simulations to check the stability of the solutions pertaining to case I, i.e. the trigonometric solution. It is worth mentioning that the numerical techniques based on the fast Fourier transform (FFT) are expensive as they require the FFT of the external source. Hence, we have used the semi-implicit Crank–Nicholson finite difference method [31] which is quite handy and unconditionally stable, to show numerically that equation (3) shows regular dynamics when the controlling function in equation (4) is applied. We numerically study the nonlinear evolution of the exact solution under small perturbation by directly simulating equation (3) together with equation (4) with initial condition $\phi(x, t = 0) = \phi(x)[1 + w]\,\mathrm{e}^{\mathrm{i}\omega t}$. This solution has been knitted on a lattice with grid size $\mathrm{d}x = 0.005$ and $\mathrm{d}t = 5.0 \times 10^{-6}$. The nonlinear evolution of the same shows the regular dynamics as depicted in figure 3 with a perturbation $w = 0.2$, although the peak of the intensity oscillates.

4. Conclusion

In conclusion, inspired by the efficacy of the recently developed GR condition, we have shown that in the presence of the control signal, the chaotic behaviour of the ac-driven CGLE can be suppressed and exact solitary wave solutions of both bright and dark can be obtained. This has been accomplished by using a fractional transformation. The reported solutions are exact solutions of the modified CGLE phase locked to the monochromatic driver provided the GR condition is satisfied. The key concept that links the situation where we have been able to suppress the chaos is based on the mutual cancellation of nonintegrable terms in view of the GR condition. By a suitable ST perturbation $F(x, t)$, we may be able to control different patterns in the ac-driven perturbed sine-Gordon equation, and obtain exact solitary wave solutions in a similar manner.

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